Vitruvian-1 Technical Report: Data-Centric Chain-of-Thought Reasoning in Multilingual Language Models

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Abstract

This report describes a data-centric approach aimed at improving multilingual chain-of-thought (CoT) reasoning in language models. We present a multistep training pipeline that includes: (i) continuous pre-training on a high-quality 120B-token multilingual corpus, (ii) supervised fine-tuning with distilled and diversified CoT data, and (iii) reinforcement learning. Below is an overview of the techniques used to build the model.

1 Continued Pretraining Corpus

1. **High-Quality Translation:** We first obtain high quality translations from a subset of the FineWeb dataset.
2. **Scaling:** Subsequently, we rely on the DeepL API to increase the volume. However, a few more steps are needed to ensure sufficient quality.

1.1 Quality Filtering with a Learned Evaluation Model

We train a quality evaluation classifier based on the Llama-3.2-1B [2] model, framing this as a next token prediction task. The training data comprises:

- **Positive examples:** High-quality translations.
- **Negative examples:** Synthetically degraded texts and plain DeepL translations.

For each document $Text_i$ in D_{CP}, we compute a quality score:

$$Q_{\text{CP}}(Text_i) = P_{M_{\text{Quality}}}(\text{high\_quality} \mid Text_i)$$

We then filter D_{CP} by selecting documents with a probability greater than $\tau_{\text{CP}}\%$ of being part to the positive examples class:

$$D'_{\text{CP}} = \{Text_i \in D_{\text{CP}} \mid Q_{\text{CP}}(Text_i) > \tau_{\text{CP}}\}$$

This ensures greater training stability than subsequent pretraining.

2 Reasoning CoT Traces

The reasoning dataset is derived from a curated internal question-answer pool D_{QA} using a two-fold process: distillation from an existing reasoning model and dynamic diversification via curriculum learning.

2.1 CoT Distillation via Rejection Sampling

For each question $x_j \in D_{\text{QA}}$, we generate $K = 8$ candidate CoT answers using the DeepSeek-R1 [1] model.

Each candidate is checked for correctness:

- **Rule-based verification:** For mathematical queries, a deterministic verifier ensures that the final answer is consistent with y_j^*.
- **Model based verification:** For general reasoning tasks, an external LLM acting as a judge ($judge$) compares the predicted and ground truth answers.

Only candidates whose final answer matches the ground truth y_j^* are retained; if no candidate passes, x_j is discarded.

2.2 Difficulty-Aware Question Diversification

To prevent overfitting and encourage exploration, we implement a difficulty-aware prompt generation strategy:

1. **Difficulty Estimation:** For each question x, we compute a difficulty score based on the chain-of-thought [7] length and final answer perplexity from a preliminary SFT checkpoint:

$$Diff(x) = \alpha \cdot Perplexity(y^* \mid x) + \beta \cdot P(\text{Incorrect} \mid Judge(x, y^*))$$

2. **Prompt Generation:** A LoRA-adapted [3] Llama-3.1-8B [2] model is fine-tuned on high-difficulty examples to generate new questions.
3. **Diversity Maximisation:** Generated answers are embedded using Sentence-BERT [5], and kernel density estimation (KDE) is applied to the embedding space. Prompts are sampled preferentially from low-density regions to ensure diversity.
4. **Answer Verification:** For each generated prompt, OpenAI's o1-preview [4] model solves the problem multiple times (typically four) with majority voting determining the final answer. Agreement with DeepSeek-R1 is required before a prompt is accepted into D_{CoT}.

3 Reasoning Supervised Fine-Tuning Data

The SFT dataset is constructed by merging the distilled and diversified CoT samples. Although the bulk of data is machine-translated, we still retain source English samples to support prevent catastrophic interference. Additionally, we leverage CoT length estimates—predicted by an auxiliary model—to stratify and balance samples across difficulty levels, following insights akin to those in curriculum learning frameworks.

4 Continued Pretraining

We initialize our base models from the Phi-4 decoder-only transformer, chosen for its parameter efficiency and few-shot capabilities. We aim to improve its capabilities and performance on a subset of European languages.

5 Reasoning Supervised Fine-Tuning

We optimize a standard SFT objective

$$L_{\text{SFT}}(\theta) = -\mathbb{E}_{(x, CoT, y^*) \sim D_{\text{SFT}}} \left[\sum_{t=1}^{|CoT|} \log P_\theta(CoT_t \mid x, CoT_{<t}) + \log P_\theta(y^* \mid x, CoT) \right]$$

6 Reinforcement Learning

6.1 Reward Design

Our RL framework uses a composite reward function that integrates:

- **Rule-Based Components:** Deterministic verifiers (e.g. unit tests for code) assess the correctness of generated responses to well-defined queries.
- **Model-Based Components:** For open-ended writing, we train a reward model to assess the stylistic quality of the text.

Additionally, we incorporate a *cosine reward* [8] term to gently bias the model toward efficient reasoning. This term is defined as:

$$R_{\text{Cosine}}(C, L_{gen}) = \begin{cases} CosFn(L_{gen}, L_{max}, r_0^c, r_L^c) & \text{if } C = 1, \\ CosFn(L_{gen}, L_{max}, r_0^w, r_L^w) & \text{if } C = 0, \\ r_e & \text{if } L_{gen} = L_{max}, \end{cases}$$

with

$$CosFn(L_{gen}, L_{max}, \eta_{min}, \eta_{max}) = \eta_{min} + \frac{1}{2}(\eta_{max} - \eta_{min}) \left(1 + \cos\left(\frac{L_{gen}\pi}{L_{max}}\right)\right).$$

This encourages shorter but accurate CoT generations, while penalizing inefficiency when answers are incorrect. This results in a more efficient use of the computational budget.

6.2 Proximal Policy Optimization

PPO [6] is used to update the model policy. The clipped surrogate objective is formulated as:

$$L^{CLIP}(\theta) = \mathbb{E}_{(s,a,A^{\pi_{old}})} \left[\min\left(\frac{\pi_\theta(a \mid s)}{\pi_{old}(a \mid s)} A^{\pi_{old}}(s,a), \text{clip}\left(\frac{\pi_\theta(a \mid s)}{\pi_{old}(a \mid s)}, 1 - \epsilon, 1 + \epsilon \right) A^{\pi_{old}}(s,a) \right) \right],$$

where s denotes the state (input prompt and CoT history), a the action (next token), and $A^{\pi_{old}}$ the advantage computed via Generalized Advantage Estimation (GAE).

7 Preliminary Evaluation

We evaluate **Vitruvian-1** on standard benchmarks to assess its performance in different domains. Although we benchmarked the model on a larger set of datasets, we couldn't complete a full evaluation due to computational constraints. Therefore, although the results are satisfactory, we decide not to release them yet to the general public to avoid the risk of overstating the model's capabilities. However, we will release a more thorough evaluation in the near future.

Benchmark	**Score**
MATH	95.5
MMLU	90.2

Table 1: Preliminary benchmark results of Vitruvian-1.

8 Future Directions and Considerations

We have intentionally omitted key information from this report that we believe is potentially relevant to our IP. This work is not intended to be a scientific article.

We see this as a promising starting point. We're expanding our computing capabilities and are already working on our next generation of models, which will come in different sizes and performance levels.

References

[1] DeepSeek-AI, Daya Guo, Dejian Yang, Haowei Zhang, Junxiao Song, Ruoyu Zhang, Runxin Xu, Qihao Zhu, Shirong Ma, Peiyi Wang, Xiao Bi, Xiaokang Zhang, Xingkai Yu, Yu Wu, Z. F. Wu, Zhibin Gou, Zhihong Shao, Zhuoshu Li, Ziyi Gao, Aixin Liu, Bing Xue, Bingxuan Wang, Bochao Wu, Bei Feng, Chengda Lu, Chenggang Zhao, Chengqi Deng, Chenyu Zhang, Chong Ruan, Damai Dai, Deli Chen, Dongjie Ji, Erhang Li, Fangyun Lin, Fucong Dai, Fuli Luo, Guangbo Hao, Guanting Chen, Guowei Li, H. Zhang, Han Bao, Hanwei Xu, Haocheng Wang, Honghui Ding, Huajian Xin, Huazuo Gao, Hui Qu, Hui Li, Jianzhong Guo, Jiashi Li, Jiawei Wang, Jingchang Chen, Jingyang Yuan, Junjie Qiu, Junlong Li, J. L. Cai, Jiaqi Ni, Jian Liang, Jin Chen, Kai Dong, Kai Hu, Kaige Gao, Kang Guan, Kexin Huang, Kuai Yu, Lean Wang, Lecong Zhang, Liang Zhao, Litong Wang, Liyue Zhang, Lei Xu, Leyi Xia, Mingchuan Zhang, Minghua Zhang, Minghui Tang, Meng Li, Miaojun Wang, Mingming Li, Ning Tian, Panpan Huang, Peng Zhang, Qiancheng Wang, Qinyu Chen, Qiushi Du, Ruiqi Ge, Ruisong Zhang, Ruizhe Pan, Runji Wang, R. J. Chen, R. L. Jin, Ruyi Chen, Shanghao Lu, Shangyan Zhou, Shanhuang Chen, Shengfeng Ye, Shiyu Wang, Shuiping Yu, Shunfeng Zhou, Shuting Pan, S. S. Li, Shuang Zhou, Shaoqing Wu, Shengfeng Ye, Tao Yun, Tian Pei, Tianyu Sun, T. Wang, Wangding Zeng, Wanjia Zhao, Wen Liu, Wenfeng Liang, Wenjun Gao, Wenqin Yu, Wentao Zhang, W. L. Xiao, Wei An, Xiaodong Liu, Xiaohan Wang, Xiaokang Chen, Xiaotao Nie, Xin Cheng, Xin Liu, Xin Xie, Xingchao Liu, Xinyu Yang, Xinyuan Li, Xuecheng Su, Xuheng Lin, X. Q. Li, Xiangyue Jin, Xiaojin Shen, Xiaosha Chen, Xiaowen Sun, Xiaoxiang Wang, Xinnan Song, Xinyi Zhou, Xianzu Wang, Xinxia Shan, Y. K. Li, Y. Q. Wang, Y. X. Wei, Yang Zhang, Yanhong Xu, Yao Li, Yao Zhao, Yaofeng Sun, Yaohui Wang, Yi Yu, Yichao Zhang, Yifan Shi, Yiliang Xiong, Ying He, Yishi Piao, Yisong Wang, Yixuan Tan, Yiyang Ma, Yiyuan Liu, Yongqiang Guo, Yuan Ou, Yuduan Wang, Yue Gong, Yuheng Zou, Yujia He, Yunfan Xiong, Yuxiang Luo, Yuxiang You, Yuxuan Liu, Yuyang Zhou, Y. X. Zhu, Yanhong Xu, Yanping Huang, Yaohui Li, Yi Zheng, Yuchen Zhu, Yunxian Ma, Ying Tang, Yukun Zha, Yuting Yan, Z. Z. Ren, Zehui Ren, Zhangli Sha, Zhe Fu, Zhean Xu, Zhenda Xie, Zhengyan Zhang, Zhewen Hao, Zhicheng Ma, Zhigang Yan, Zhiyu Wu, Zihui Gu, Zijia Zhu, Zijun Liu, Zilin Li, Ziwei Xie, Ziyang Song, Zizheng Pan, Zhen Huang, Zhipeng Xu, Zhongyu Zhang, and Zhen Zhang. Deepseek-r1: Incentivizing reasoning capability in llms via reinforcement learning, 2025.

[2] Aaron Grattafiori, Abhimanyu Dubey, Abhinav Jauhri, Abhinav Pandey, Abhishek Kadian, Ahmad Al-Dahle, Aiesha Letman, Akhil Mathur, Alan Schelten, Alex Vaughan, Amy Yang, Angela Fan, Anirudh Goyal, Anthony Hartshorn, Aobo Yang, Archi Mitra, Archie Sravankumar, Artem Korenev, Arthur Hinsvark, Arun Rao, Aston Zhang, Aurelien Rodriguez, Austen Gregerson, Ava Spataru, Baptiste Roziere, Bethany Biron, Binh Tang, Bobbie Chern, Charlotte Caucheteux, Chaya Nayak, Chloe Bi, Chris Marra, Chris McConnell, Christian Keller, Christophe Touret, Chunyang Wu, Corinne Wong, Cristian Canton Ferrer, Cyrus Nikolaidis, Damien Allonsius, Daniel Song, Danielle Pintz, Danny Livshits, Danny Wyatt, David Esiobu, Dhruv Choudhary, Dhruv Mahajan, Diego Garcia-Olano, Diego Perino, Dieuwke Hupkes, Egor Lakomkin, Ehab AlBadawy, Elina Lobanova, Emily Dinan, Eric Michael Smith, Filip Radenovic, Francisco Guzmán, Frank Zhang, Gabriel Synnaeve, Gabrielle Lee, Georgia Lewis Anderson, Govind Thattai, Graeme Nail, Gregoire Mialon, Guan Pang, Guillem Cucurell, Hailey Nguyen, Hannah Korevaar, Hu Xu, Hugo Touvron, Iliyan Zarov, Imanol Arrieta Ibarra, Isabel Kloumann, Ishan Misra, Ivan Evtimov, Jack Zhang, Jade Copet, Jaewon Lee, Jan Geffert, Jana Vranes, Jason Park, Jay Mahadeokar, Jeet Shah, Jelmer van der Linde, Jennifer Billock, Jenny Hong, Jenya Lee, Jeremy Fu, Jianfeng Chi, Jianyu Huang, Jiawen Liu, Jie Wang, Jiecao Yu, Joanna Bitton, Joe Spisak, Jongsoo Park, Joseph Rocca, Joshua Johnstun, Joshua Saxe, Junteng Jia, Kalyan Vasuden Alwala, Karthik Prasad, Kartikeya Upasani, Kate Plawiak, Ke Li, Kenneth Heafield, Kevin Stone, Khalid El-Arini, Krithika Iyer, Kshitiz Malik, Kuenley Chiu, Kunal Bhalla, Kushal Lakhotia, Lauren Rantala-Yeary, Laurens van der Maaten, Lawrence Chen, Liang Tan, Liz Jenkins, Louis Martin, Lovish Madaan, Lubo Malo, Lukas Blecher, Lukas Landzaat, Luke de Oliveira, Madeline Muzzi, Mahesh Pasupuleti, Mannat Singh, Manohar Paluri, Marcin Kardas, Maria Tsimpoukelli, Mathew Oldham, Mathieu Rita, Maya Pavlova, Melanie Kambadur, Mike Lewis, Min Si, Mitesh Kumar Singh, Mona Hassan, Naman Goyal, Narjes Torabi, Nikolay Bashlykov, Nikolay Bogoychev, Niladri Chatterji, Ning Zhang, Olivier Duchenne, Onur Çelebi, Patrick Alrassy, Pengchuan Zhang, Pengwei Li, Petar Vasic, Peter Weng, Prajjwal Bhargava, Pratik Dubal, Praveen Krishnan, Punit Singh Koura, Puxin Xu, Qing He, Qingxiao Dong, Ragavan Srinivasan, Raj Ganapathy, Ramon Calderer, Ricardo Silveira Cabral, Robert Stojnic, Roberta Raileanu, Rohan Maheswari, Rohit Girdhar, Rohit Patel, Romain Sauvestre, Ronnie Polidoro, Roshan Sumbaly, Ross Taylor, Ruan Silva, Rui Hou, Rui Wang, Saghar Hosseini, Sahana Chennabasappa, Sanjay Singh, Sean Bell, Seo-

hyun Sonia Kim, Sergey Edunov, Shaoliang Nie, Sharan Narang, Sharath Raparthy, Sheng Shen, Shengye Wan, Shruti Bhosale, Shun Zhang, Simon Vandenhende, Soumya Batra, Spencer Whitman, Sten Sootla, Stephane Collot, Suchin Gururangan, Sydney Borodinsky, Tamar Herman, Tara Fowler, Tarek Sheasha, Thomas Georgiou, Thomas Scialom, Tobias Speckbacher, Todor Mihaylov, Tong Xiao, Ujjwal Karn, Vedanuj Goswami, Vibhor Gupta, Vignesh Ramanathan, Viktor Kerkez, Vincent Gonguet, Virginie Do, Vish Vogeti, Vítor Albiero, Vladan Petrovic, Weiwei Chu, Wenhan Xiong, Wenyin Fu, Whitney Meers, Xavier Martinet, Xiaodong Wang, Xiaofang Wang, Xiaoqing Ellen Tan, Xide Xia, Xinfeng Xie, Xuchao Jia, Xuewei Wang, Yaelle Goldschlag, Yashesh Gaur, Yasmine Babaei, Yi Wen, Yiwen Song, Yuchen Zhang, Yue Li, Yuning Mao, Zacharie Delpierre Coudert, Zheng Yan, Zhengxing Chen, Zoe Papakipos, Aaditya Singh, Aayushi Srivastava, Abha Jain, Adam Kelsey, Adam Shajnfeld, Adithya Gangidi, Adolfo Victoria, Ahuva Goldstand, Ajay Menon, Ajay Sharma, Alex Boesenberg, Alexei Baevski, Allie Feinstein, Amanda Kallet, Amit Sangani, Amos Teo, Anam Yunus, Andrei Lupu, Andres Alvarado, Andrew Caples, Andrew Gu, Andrew Ho, Andrew Poulton, Andrew Ryan, Ankit Ramchandani, Annie Dong, Annie Franco, Anuj Goyal, Aparajita Saraf, Arkabandhu Chowdhury, Ashley Gabriel, Ashwin Bharambe, Assaf Eisenman, Azadeh Yazdan, Beau James, Ben Maurer, Benjamin Leonhardi, Bernie Huang, Beth Loyd, Beto De Paola, Bhargavi Paranjape, Bing Liu, Bo Wu, Boyu Ni, Braden Hancock, Bram Wasti, Brandon Spence, Brani Stojkovic, Brian Gamido, Britt Montalvo, Carl Parker, Carly Burton, Catalina Mejia, Ce Liu, Changhan Wang, Changkyu Kim, Chao Zhou, Chester Hu, Ching-Hsiang Chu, Chris Cai, Chris Tindal, Christoph Feichtenhofer, Cynthia Gao, Damon Civin, Dana Beaty, Daniel Kreymer, Daniel Li, David Adkins, David Xu, Davide Testuggine, Delia David, Devi Parikh, Diana Liskovich, Didem Foss, Dingkang Wang, Duc Le, Dustin Holland, Edward Dowling, Eissa Jamil, Elaine Montgomery, Eleonora Presani, Emily Hahn, Emily Wood, Eric-Tuan Le, Erik Brinkman, Esteban Arcaute, Evan Dunbar, Evan Smothers, Fei Sun, Felix Kreuk, Feng Tian, Filippos Kokkinos, Firat Ozgenel, Francesco Caggioni, Frank Kanayet, Frank Seide, Gabriela Medina Florez, Gabriella Schwarz, Gada Badeer, Georgia Swee, Gil Halpern, Grant Herman, Grigory Sizov, Guangyi, Zhang, Guna Lakshminarayanan, Hakan Inan, Hamid Shojanazeri, Han Zou, Hannah Wang, Hanwen Zha, Haroun Habeeb, Harrison Rudolph, Helen Suk, Henry Aspegren, Hunter Goldman, Hongyuan Zhan, Ibrahim Damlaj, Igor Molybog, Igor Tufanov, Ilias Leontiadis, Irina-Elena Veliche, Itai Gat, Jake Weissman, James Geboski, James Kohli, Janice Lam, Japhet Asher, Jean-Baptiste Gaya, Jeff Marcus, Jeff Tang, Jennifer Chan, Jenny Zhen, Jeremy Reizenstein, Jeremy Teboul, Jessica Zhong, Jian Jin, Jingyi Yang, Joe Cummings, Jon Carvill, Jon Shepard, Jonathan McPhie, Jonathan Torres, Josh Ginsburg, Junjie Wang, Kai Wu, Kam Hou U, Karan Saxena, Kartikay Khandelwal, Katayoun Zand, Kathy Matosich, Kaushik Veeraraghavan, Kelly Michelena, Keqian Li, Kiran Jagadeesh, Kun Huang, Kunal Chawla, Kyle Huang, Lailin Chen, Lakshya Garg, Lavender A, Leandro Silva, Lee Bell, Lei Zhang, Liangpeng Guo, Licheng Yu, Liron Moshkovich, Luca Wehrstedt, Madian Khabsa, Manav Avalani, Manish Bhatt, Martynas Mankus, Matan Hasson, Matthew Lennie, Matthias Reso, Maxim Groshev, Maxim Naumov, Maya Lathi, Meghan Keneally, Miao Liu, Michael L. Seltzer, Michal Valko, Michelle Restrepo, Mihir Patel, Mik Vyatskov, Mikayel Samvelyan, Mike Clark, Mike Macey, Mike Wang, Miquel Jubert Hermoso, Mo Metanat, Mohammad Rastegari, Munish Bansal, Nandhini Santhanam, Natascha Parks, Natasha White, Navyata Bawa, Nayan Singhal, Nick Egebo, Nicolas Usunier, Nikhil Mehta, Nikolay Pavlovich Laptev, Ning Dong, Norman Cheng, Oleg Chernoguz, Olivia Hart, Omkar Salpekar, Ozlem Kalinli, Parkin Kent, Parth Parekh, Paul Saab, Pavan Balaji, Pedro Rittner, Philip Bontrager, Pierre Roux, Piotr Dollar, Polina Zvyagina, Prashant Ratanchandani, Pritish Yuvraj, Qian Liang, Rachad Alao, Rachel Rodriguez, Rafi Ayub, Raghotham Murthy, Raghu Nayani, Rahul Mitra, Rangaprabhu Parthasarathy, Raymond Li, Rebekkah Hogan, Robin Battey, Rocky Wang, Russ Howes, Ruty Rinott, Sachin Mehta, Sachin Siby, Sai Jayesh Bondu, Samyak Datta, Sara Chugh, Sara Hunt, Sargun Dhillon, Sasha Sidorov, Satadru Pan, Saurabh Mahajan, Saurabh Verma, Seiji Yamamoto, Sharadh Ramaswamy, Shaun Lindsay, Shaun Lindsay, Sheng Feng, Shenghao Lin, Shengxin Cindy Zha, Shishir Patil, Shiva Shankar, Shuqiang Zhang, Shuqiang Zhang, Sinong Wang, Sneha Agarwal, Soji Sajuyigbe, Soumith Chintala, Stephanie Max, Stephen Chen, Steve Kehoe, Steve Satterfield, Sudarshan Govindaprasad, Sumit Gupta, Summer Deng, Sungmin Cho, Sunny Virk, Suraj Subramanian, Sy Choudhury, Sydney Goldman, Tal Remez, Tamar Glaser, Tamara Best, Thilo Koehler, Thomas Robinson, Tianhe Li, Tianjun Zhang, Tim Matthews, Timothy Chou, Tzook Shaked, Varun Vontimitta, Victoria Ajayi, Victoria Montanez, Vijai Mohan, Vinay Satish Kumar, Vishal Mangla, Vlad Ionescu, Vlad Poenaru, Vlad Tiberiu Mihailescu, Vladimir Ivanov, Wei Li, Wenchen Wang, Wenwen Jiang, Wes Bouaziz, Will Constable, Xiaocheng Tang, Xiaojian Wu, Xiaolan Wang, Xilun Wu, Xinbo Gao, Yaniv Kleinman, Yanjun Chen, Ye Hu, Ye Jia, Ye Qi, Yenda Li, Yilin Zhang, Ying

Zhang, Yossi Adi, Youngjin Nam, Yu, Wang, Yu Zhao, Yuchen Hao, Yundi Qian, Yunlu Li, Yuzi He, Zach Rait, Zachary DeVito, Zef Rosnbrick, Zhaoduo Wen, Zhenyu Yang, Zhiwei Zhao, and Zhiyu Ma. The llama 3 herd of models, 2024.

[3] Edward J Hu, Yelong Shen, Phillip Wallis, Zeyuan Allen-Zhu, Yuanzhi Li, Shean Wang, Lu Wang, and Weizhu Chen. Lora: Low-rank adaptation of large language models. *arXiv preprint arXiv:2106.09685*, 2021.

[4] Aaron Jaech, Adam Kalai, Adam Lerer, Adam Richardson, Ahmed El-Kishky, Aiden Low, Alec Helyar, Aleksander Madry, Alex Beutel, Alex Carney, et al. Openai o1 system card. *arXiv preprint arXiv:2412.16720*, 2024.

[5] Nils Reimers and Iryna Gurevych. Sentence-bert: Sentence embeddings using siamese bert-networks, 2019.

[6] John Schulman, Filip Wolski, Prafulla Dhariwal, Alec Radford, and Oleg Klimov. Proximal policy optimization algorithms. *arXiv preprint arXiv:1707.06347*, 2017.

[7] Jason Wei, Xuezhi Wang, Dale Schuurmans, Maarten Bosma, Ed H. Chi, Quoc Le, and Denny Zhou. Chain of thought prompting elicits reasoning in large language models. *CoRR*, abs/2201.11903, 2022.

[8] Edward Yeo, Yuxuan Tong, Morry Niu, Graham Neubig, and Xiang Yue. Demystifying long chain-of-thought reasoning in llms, 2025.